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Continuity and its Application to Reliability Engineering**

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## Construction of Response Surface with Higher Order Continuity and its Application to Reliability Engineering

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### Abstract

The usefulness of piecewise polynomials with  $C^1$  and  $C^2$  derivative continuity for response surface construction method is examined. A Moving Least Squares (MLS) method is developed and compared with four other interpolation methods, including kriging. First the selected methods are applied and compared with one another in a two-design variables problem with a known theoretical response function. Next the methods are tested in a four-design variables problem from a reliability-based design application. In general the piecewise polynomial with higher order derivative continuity methods produce less error in the response prediction. The MLS method was found to be superior for response surface construction among the methods evaluated.

### Introduction

Structural reliability engineering analysis involves determination of probability of structural failure taking into account the uncertainty in the geometric parameters, material properties and loading conditions<sup>1</sup>. The uncertain quantities are treated as random variables with known probability distributions in probabilistic analysis. In these analyses, for each set of random variables a Monte Carlo simulation is performed to determine the probability of failure of the structure<sup>2</sup>. Monte Carlo simulations require large number of simulations and hence, are computationally expensive and also require large amounts of analyses time. In order to alleviate some of the problems associated with

Monte Carlo simulation, approximate methods such as First Order Reliability Method (FORM) and Second Order Reliability Method (SORM) were developed to minimize the number of simulations<sup>2</sup>. Even with these approximate methods, the computational effort required to perform a structural reliability analysis can be very high. Thus, there is a need to develop methods that are accurate and yet minimize the computational time.

Response surface functions are often used as simple and inexpensive replacements for computationally expensive structural analyses in reliability methods. In the response surface method, a surface is fit to interpolate data that are generated by performing structural simulations at selected points in the parameter space. The response surface is then used to evaluate the structural response at other points in parameter space. For example, the polynomial regression response surface method<sup>3</sup> is widely used in structural optimization. The number of data points necessary to fully sample the parameter space increases exponentially as the number of random variables or design variables increases. In theory it is possible to find an interpolating polynomial of sufficiently high order to pass through all the data points in the parameter space. However, for such interpolating polynomials as the number of data points increases, the surface generated tends to exhibit oscillations or "wiggleness" between the data points. For this reason the polynomials are generally limited to first and second order to represent the response surface, and Least Squares regression is used to best fit the data points. Many other methods use piecewise polynomials instead of a single global polynomial to represent the entire

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parameter space. The piecewise polynomials are generally limited to  $C^0$  continuous surfaces. In  $C^0$  continuous surfaces, the function values are continuous everywhere (across piecewise polynomials), but first derivative (slope) continuity and higher order derivative continuity are not guaranteed. Recently Romero et al.<sup>4,5</sup> implemented a Progressive-Lattice-Sampling [PLS] method based on finite element interpolation ( $C^0$  globally continuous) to construct the response surfaces. However, response surface construction methods using piecewise polynomials with  $C^1$  continuity (function values and slopes are continuous) and with  $C^2$  continuity (function values, slopes and second derivatives are continuous) are rare in the literature.

The purpose of the present paper is to study the usefulness of piecewise polynomials (with higher order derivative continuity) on response surface generation and its application to reliability engineering. The Minimum Norm Network (MNN) developed in references 6-10 adopted to construct a response surface with  $C^1$  and  $C^2$  continuity in two design variables. A new enhanced  $C^2$  MNN method is developed to reproduce polynomials up to third degrees exactly. Also a Moving Least Squares (MLS) method is developed to reproduce  $C^1$  and  $C^2$  continuous response surfaces for arbitrary number of design variables. The selected interpolation methods are tested in a two variables problem and results are compared. Finally a four-variable example from a reliability application" is presented to demonstrate the effectiveness of the MLS method. The MLS method is compared with least squares polynomial and the kriging methods on this problem.

### Interpolation Methods

The accuracy of the response surface in representing the behavior of the actual system largely depends upon the interpolating method used for its generation. Brief introductions to the five interpolating methods used in the current analysis are presented in this section.

#### 1. Global Least Squares Method (GLS)

The GLS methods are generally known as polynomial regression methods and are widely used in the literature [3]. The GLS methods are used to create response surface functions from a set of sampling points. For

example, a quadratic polynomial with  $NDV$  design variables has the form

$$\hat{y}(X) = a_0 + \sum_{j=1}^{NDV} a_j X_j + \sum_{j=1, i=j}^{NDV} b_{ij} X_i X_j \quad (1)$$

Where  $\hat{y}$  is the approximated value of the target function at the point in the parameter space having coordinates  $(X_1, X_2, \dots, X_{NDV})$ , and  $a_0, a_j$ , and  $b_{ij}$  are the unknown constant coefficients. The unknown coefficients are determined by a regression procedure. Most commonly, the method of least squares is used to determine the coefficients that minimize the error of the approximation at the sampling points<sup>3</sup>. Since a single polynomial is used to represent the entire parametric space, the method is termed here as the Global Least Squares (GLS) method. In the present study, the GLS method is limited to the quadratic polynomial given by Equation (1).

#### 2. Kriging

Kriging is an interpolation method that originated in the geostatistics community. Kriging uses the properties of the spatial correlation among the data samples. In arriving at an interpolated value at some point in the parameter space, kriging more heavily weights data samples that are "nearby" rather than giving all data samples equal weight. This is achieved by setting mean residual error to zero and also minimizing the variance of the errors. The final equations for kriging are given below from reference 12 for  $N$  sampling points and  $NDV$  design variables:

The estimated value of  $\hat{y}$  in kriging is obtained from

$$\hat{y} = \hat{\beta} + r^T R^{-1} (Y - \hat{\beta} f) \quad (2)$$

where  $Y$  is the column vector of known function values at the  $N$  sampling points,  $\hat{\beta}$  is a constant to be determined,  $R$  is correlation matrix obtained for an  $i^{th}$  row and  $j^{th}$  column from the correlation function as

$$R(X^i, X^j) = \exp \left[ -\theta \sum_{k=1}^{NDV} |X_k^i - X_k^j|^2 \right], \quad (3)$$

$r$  is the column vector of length  $N$  obtained from

$$r^T = \{R(X, X^1), R(X, X^2), \dots, R(X, X^N)\}^T \quad (4)$$

and  $f$  is vector of length  $N$ , with all elements in the vector set to unity as

$$f^T = \{1, 1, 1, \dots, 1\}^T \quad (5)$$

The unknown  $\hat{\beta}$  in Equation (2) can be obtained from

$$\hat{\beta} = (f^T R^{-1} f)^{-1} f^T R^{-1} Y \quad (6)$$

The Maximum Likelihood Estimate (MLE) for the unknown quantity  $\theta$  in Equations (3) is obtained from a one-dimensional maximization problem defined by

$$\text{MLE} = \max \left( \frac{-1}{2} \right) \left[ N \ln(\hat{\sigma}^2) + \ln|R| \right] \quad 0 \leq \theta \leq \infty \quad (7)$$

Where

$$\hat{\sigma}^2 = \frac{(Y - \beta f)^T R^{-1} (Y - \beta f)}{N} \quad (8)$$

Estimation of  $\theta$  in the one dimensional optimization problem is the critical step in the kriging method. The kriging method used in this study produces a  $C^2$  continuous interpolating function over the entire parameter space.

### **3. Minimum Norm Network (MNN):**

The MNN method is a piecewise interpolation scheme that can produce  $C^1$  and  $C^2$  continuous surfaces. Complete details about the method can be found in references 6-10, only a brief description of the method is presented here. The MNN method requires the function values at the arbitrary sampling points. The procedure to construct a MNN can be described in the following three steps:

1. The given sampling points are triangulated such that each point lies at a vertex of the triangles.
2. The derivatives are generated at the vertices of the triangles from known function values. On each **edge** of the triangles an interpolating polynomial is fit in terms of the known function values and unknown partial

derivatives at the vertices. The unknown partial derivatives are determined by enforcing continuity of all edges meeting at a vertex by minimizing a functional (second order for 2 parameters). The union of all the edges of all the triangles forms the MNN network. At the end of the second step, function values and its derivatives are known at the vertices of the triangles in the network.

3. The interpolation is extended inside the triangles in terms of the function values and its derivatives at the vertices of the triangle. The interpolation inside the triangles is achieved using triangular blending functions forming a piecewise continuous surface. The blending function is constructed in such way that it produces  $C^1$  and  $C^2$  continuous surface with its neighboring triangles.

A response surface with  $C^1$  and  $C^2$  continuity is produced using the method developed in references 6-9 and used for the interface element development in reference 10. The MNN from reference 8 produces a  $C^2$  continuous surface in the limit. Hence, a new "enhanced  $C^2$ " method is developed here to produce  $C^3$  continuity (partial derivatives up to third degrees continuous) in step 2 of MNN network and uses the same blending function as that of  $C^2$  in step 3.

At present the MNN method is available only for two design variables (dimensions). Extending the MNN to more than two variables is difficult. However, these methods are included in this study, since, they are the only methods that provide piecewise polynomial interpolation. Also these methods can produce a  $C^2$  continuous surface that passes through all the sampling points for arbitrarily oriented data points (i.e., not in a structured grid).

### **4. Moving Least Squares Method (MLS):**

The Moving Least Squares (MLS) method is widely used in meshless methods<sup>13,14</sup>. Recently the MLS method has been successfully applied for response surface generation in the context of optimization in reference 15. A MLS method is developed here for an

arbitrary number of design variables. The method is briefly discussed here:

The MLS approximation for the estimated value  $\hat{y}(x)$  can be written as

$$\hat{y}\{X\} = p^T\{X\} a_m(X) \quad (9)$$

where  $p^T\{X\} = [p_1\{X\}, p_2\{X\}, \dots, p_m\{X\}]$  is a polynomial basis function of order  $m$  used in the MLS interpolation.  $a_m(X)$  is a vector containing coefficients  $a_j(X)$ ,  $j = 1, 2, \dots, m$ , which are functions of the spatial coordinates. For example, for two design variables,

$$\begin{aligned} p^T\{X\} &= [1, X_1, X_2] \text{ Linear basis function; } m = 3 \\ p^T\{X\} &= [1, X_1, X_2, X_1^2, X_1X_2, X_2^2] \text{ Quadratic basis function; } m = 6 \end{aligned} \quad (10)$$

The unknown coefficients  $a_m$  can be determined using the weighted least squares error norm  $J(X)$  at the  $N$  sampling points

$$\begin{aligned} J(X) &= \sum_{i=1}^N w_i(X) [p^T(X_i) a_m(X) - Y]^2 \\ &= [P a_m(X) - Y]^T \cdot W \cdot [P a_m(X) - Y] \end{aligned} \quad (11)$$

where  $w_i(X)$  is weight function associated with node  $i$ , whose value is nonzero only in the support or influence domain of the node  $X_i$  (usually a sphere of radius  $R_i$ ). The matrices  $P$  and  $W$  are defined as

$$P = \begin{Bmatrix} p^T(X_1) \\ p^T(X_2) \\ \dots \\ p^T(X_N) \end{Bmatrix}_{N \times m} \quad (12)$$

$$W = \begin{bmatrix} w_1(X) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & w_N(X) \end{bmatrix}_{N \times N \text{ diagonal matrix}} \quad (13)$$

and

$$Y = \{y_1, y_2, \dots, y_N\} \quad (14)$$

Minimizing the norm  $J(X)$  in Equation (10) with respect to  $a_m(X)$  leads to the following linear relation between  $a_m(X)$  and  $y$

$$A(X) a_m(X) = B(X) Y \quad (15)$$

where the matrices  $A(X)$  and  $B(X)$  are defined by

$$A(X) = P^T W P = \sum_{i=1}^N w_i(X) p(X_i) p^T(X_i) \quad (16)$$

$$\begin{aligned} B(X) &= P^T W \\ &= [w_1(X)p(X_1) \quad w_2(X)p(X_2) \quad \dots \quad w_N(X)p(X_N)] \end{aligned} \quad (17)$$

The unknown coefficients  $a_m(X)$  can be obtained by solving Equation (15), which results in

$$a_m(X) = A^{-1}(X) B(X) Y \quad (18)$$

Substituting the unknown coefficient from Equation (18) into the Equation (9) leads the MLS interpolation of the  $\hat{y}$  as

$$\hat{y} = p^T(X) A^{-1}(X) B(X) Y \quad (19)$$

The MLS approximation given in Equation (18) is well defined only when the matrix  $A$  is non-singular. This is true only if there are at least  $n$  sampling points in the influence domain of a node  $X_i$  such that  $n \geq m$ . For example, for a one-dimensional case with a linear basis function ( $m = 2$ ), the value of  $n$  should be  $\geq 2$ . For a quadratic basis function in a two-dimensional case the value of  $n$  should be  $\geq 6$ .

Except for the weight function  $w_i(X)$ , all other quantities in the MLS approximation are well defined. As already mentioned, the weight function is non-zero only in the influence domain of a node  $i$ , and equal to zero outside the influence domain. In the present study, the influence domain is assumed to be of a sphere with radius  $l_i$ . The radius  $l_i$  must be large enough to contain at least  $m$  nodes in each direction of the parametric space. The weight function is selected such that its value goes from unity at the center of the influence domain to zero at the boundary and outside the influence domain. This property of the weight function makes MLS a local approximation compared to the GLS approximation traditionally used to represent the entire domain by a single function. It may

be noted that in the MLS method for every new interpolation point ( $\hat{y}$ ) Equation (18) is formed and solved.

In this paper, three spline functions with  $C^1, C^2$ , and  $C^3$  continuity are used as weight function

For  $C^1$ :

$$w_i(x) = \begin{cases} 1 - 3\rho_i^2 + 2\rho_i^3 & 0 \leq \rho_i \leq 1 \\ 0 & \rho_i > 1 \end{cases} \quad (20)$$

For  $C^2$

$$w_i(x) = \begin{cases} 1 - 10\rho_i^3 + 15\rho_i^4 - 6\rho_i^5 & 0 \leq \rho_i \leq 1 \\ 0 & \rho_i > 1 \end{cases} \quad (21)$$

For  $C^3$

$$w_i(x) = \begin{cases} 1 - 35\rho_i^4 + 84\rho_i^5 - 70\rho_i^6 + 20\rho_i^7 & 0 \leq \rho_i \leq 1 \\ 0 & \rho_i > 1 \end{cases} \quad (22)$$

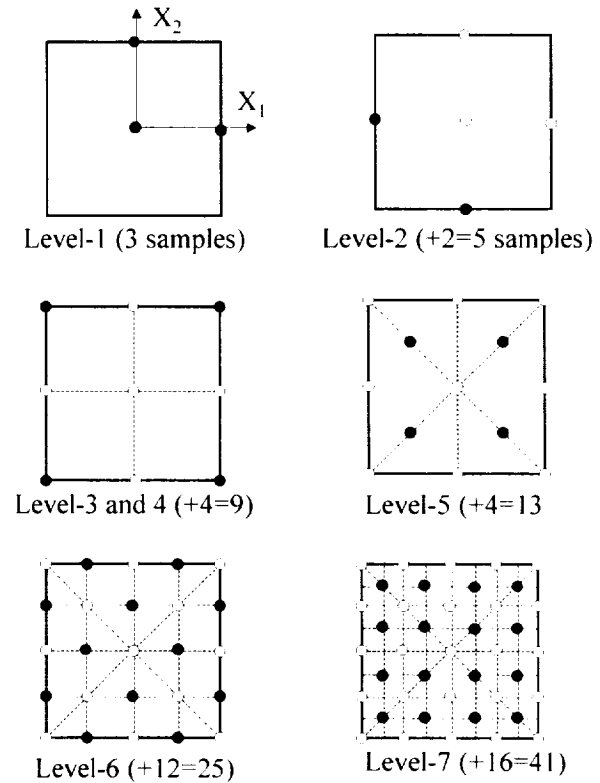
where  $\rho_i = \frac{|X_i - X|}{l_i}$  is the normalized distance, from the center of the influence domain ( $X_i$ ) and a general point  $X$ .

The smoothness of MLS approximation is controlled by both the weight and basis functions. The precision (continuity) of MLS interpolation will be equal to the minimum precision of the weight and basis function.

### 5. Piecewise Finite-Element (FE) Interpolation

Piecewise finite element (FE) interpolation was originally used<sup>4</sup> in conjunction with Progressive Lattice Sampling (PLS). PLS is discussed later in this paper. The arrangement of samples in each PLS Level allows the parameter space to be subdivided into a regular pattern of adjacent polygons, which for, two-dimensions, results in triangular and quadrilateral finite elements. For the 2-D parameter space, a combination of 3-, 4-, and 6-node triangles and 9-node quadrilaterals exist at the various PLS Levels. Low-order polynomial

interpolation is applied over each element. The collection of all the elements together creates a locally compliant  $C^0$  continuous function over the parameter space. A detailed discussion of the implementation of the FE interpolation technique used here is presented in reference 4.



**Figure 1. Progressive Lattice Sampling Points**

### Progressive Lattice Sampling (PLS) Experimental Design:

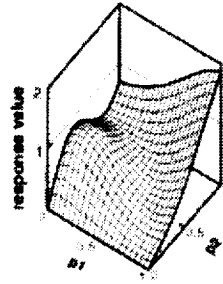
The selection of sampling points plays a major role in the accuracy of a response surface. There are many schemes available in the literature (see reference 3). Romero et al.<sup>4</sup> used Progressive Lattice Sampling (PLS) incremental experimental design sequence as shown in Figure 1 for two variables  $X_1$  and  $X_2$ . In this example, the square represents the parameter space of  $X_1$  and  $X_2$ . Level 1 of the design consists of three samples, with one sample in the center of the parameter space and two other samples along the boundary. For  $n$  parameters, Level 1 requires  $n+1$  samples. Level 2 adds  $n$  samples to complete a  $2n+1$  "simple-quadratic" layout. Level 3 adds a  $2^n$  factorial design.

Level 4 adds a Box-Behnken design to complete an over all 3<sup>rd</sup> full factorial design. (In 2-D, Levels 3 and 4 have the same layout.) Level 5 adds a sub-scaled 2<sup>nd</sup> factorial design as shown in the figure. Level 6 adds the appropriate samples to complete a 5<sup>th</sup> full factorial design. Level 7 adds a sub-scaled 4<sup>th</sup> full-factorial design in the interior of the parameter space as shown. The strength of PLS is that it provides an efficient way to add sample sites that leverage previous samples so that uniform distribution of the samples over the parameter space is maintained. The PLS design points will be used as sampling points in all the fitting and interpolation schemes in the present study.

### Application Problem

#### Two-Variables Problem:

First the described interpolation methods are applied to a two-design variables problem selected from reference 4. Target response function for the two-variables is shown in Figure 2



**Figure 2. Target Response Function for Two-Variables Problem**

This response function is defined as:

$$\begin{aligned} \text{response}(X_1, X_2) &= \left[ 0.8r + 0.35 \sin\left(\frac{2.4\pi r}{\sqrt{2}}\right) \right] [1.5 \sin(1.3\theta)] \\ &\text{on the domain } 0 \leq X_1 \leq 1, \quad 0 \leq X_2 \leq 1 \end{aligned} \quad (23)$$

$$\text{with } r = \sqrt{X_1^2 + X_2^2}, \theta = \arctan\left(\frac{X_2}{X_1}\right)$$

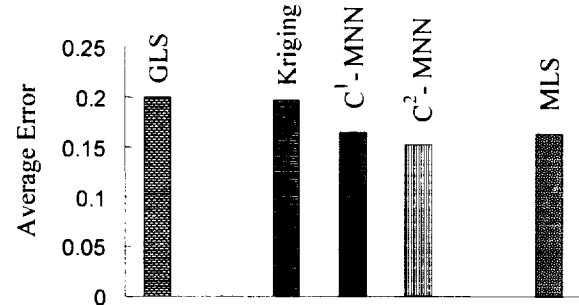
Exact data values of this function are obtained at the PLS sampling points shown in Figure 1. The response surface is generated for each of the various PLS levels in Figure 1. The response surface is then used to interpolate the value at any other point.

To examine the fitting performance (within the PLS frame work) of the various response surface construction methods, a global measure of average error is defined as follows:

$$\text{Average Error} = \frac{\sum_{i=1}^N |(exact)_i - (predicted)_i|}{N} \quad (24)$$

Where “exact” in the summation comes from the evaluation of the exact function. The predicted values in the summation come from the response surface approximation at  $N$  interpolated points. For this example  $N$  is set to equal to 441 and selected from equally spaced points on a 21x21 square grid overlaid on the domain. Earlier experience in reference 5 indicates the 21x21 grid appears to be sufficiently dense to achieve adequate representation of the target and approximate functions.

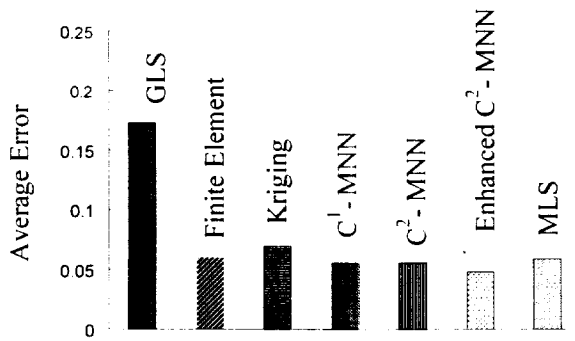
Four levels with 9, 13, 25 and 41 points were selected for comparison. For this example problem, the average errors shown in the following figures for kriging and finite element interpolation are taken from reference 5.



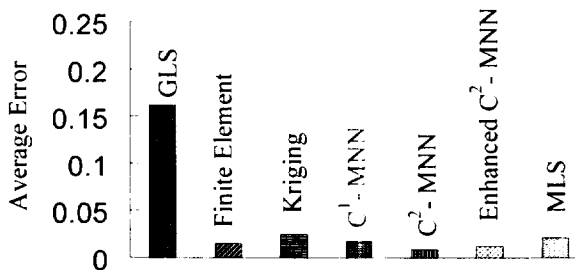
**Figure 3. Average Errors in Two-Variables Problem in Level-4 with 9 Sampling Points**

Figures 3 to 6 show the average errors from levels 4 to 7. Since 9 points are not enough to fit a quadratic basis function in MLS interpolation, a linear basis function is used. Similarly the 9 points are not sufficient to fit the Enhanced C<sup>2</sup>-MNN and hence it is not shown in Figure 3. Finite element results for level-4 were not available and are not shown in Figure 3.

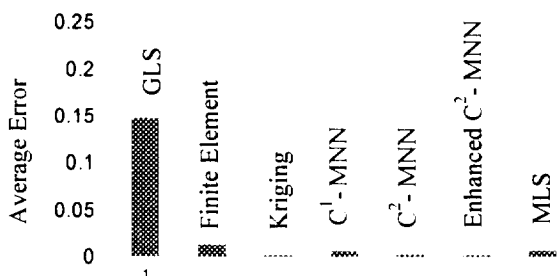




**Figure 4. Average Errors in Two-Variables Problem in Level-5 with 13 Sampling Points**



**Figure 5. Average Errors in Two-Variables Problem in Level-6 with 25 Sampling Points**



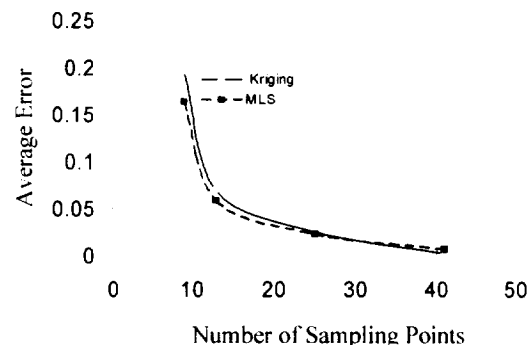
**Figure 6. Average Errors in Two-Variables Problem in Level-7 with 41 Sampling Points**

From Figures 3 to 6, the following observations can be made:

1. All the interpolation methods show reduced average errors as the number of sampling points increases.

2. At lower number of sample points, level-4, all the methods produce almost the same average error.
3. In all the levels the GLS method consistently performs poorly. This implies that the GLS method should be avoided for response surface generation.
4. All the piecewise interpolation methods including the finite element interpolation methods perform better at all levels.
5. The locally weighted methods, kriging and MLS, perform as well as the piecewise interpolation method like the MNN method.
6. The C<sup>1</sup>-MNN method performed as well as C<sup>2</sup>-MNN method at all levels.
7. All the methods with higher order derivative continuity produce less error at all levels, and
8. The kriging and Enhanced C<sup>2</sup>-MNN methods produce the least error at level 7, closely followed by MLS method.

Since it is easy to extend the kriging and MLS methods for arbitrary number of variables, the two methods are compared separately in Figure 7. It can be concluded from Figure 7, that MLS consistently produced marginally smaller errors than kriging at all levels except level-7 with 41 sampling points. However, since the difference in error between the two methods is small, there is no clear advantage of one method over the other.

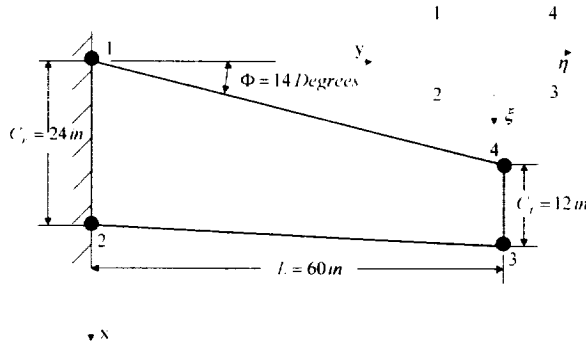


**Figure 7. Kriging and MLS Methods Comparison for Two-Variables Problem**

#### **Four-Variables Problem:**

The next example problem is taken from reliability-based design of a metal, plate-like wing to meet strength and flutter requirements<sup>11</sup>. The selected plate-like wing configuration is shown in figure 9.

The dimensions used for wingspan ( $L$ ), wing root chord ( $C_R$ ), tip chord ( $C_t$ ), and sweep of the leading edge ( $\Phi$ ) are also show in Figure 9. The modulus of elasticity is  $10 \times 10^6 \text{ psi}$  and Poisson's ratio is 0.30. The wing is clamped at the root and subjected to a uniform pressure of  $1 \text{ psi}$ .



**Figure 8. Dimension of Metal Plate-Like Wing**

The thickness distribution along the span of the wing is assumed as bi-linear and can be defined in terms of the thicknesses of the corner nodes 1 to 4 (see Figure 8) as

$$t(\xi, \eta) = c_1 + c_2\xi + c_3\eta + c_4\xi\eta \quad (25)$$

where

$$c_1 = \frac{(t_1 + t_2 + t_3 + t_4)}{4} \quad (26)$$

$$c_2 = \frac{(-t_1 + t_2 + t_3 - t_4)}{4} \quad (27)$$

$$c_3 = \frac{(-t_1 - t_2 + t_3 + t_4)}{4} \quad (28)$$

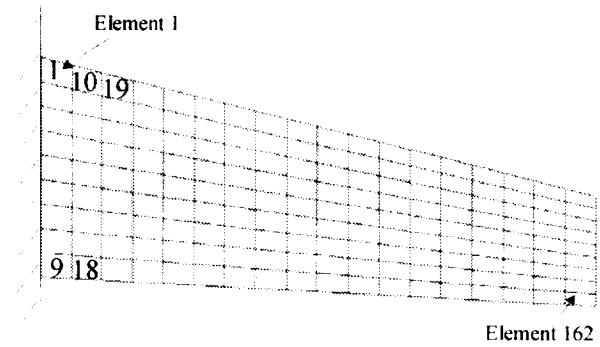
$$c_4 = \frac{(t_1 - t_2 + t_3 - t_4)}{4} \quad (29)$$

The equation relating the  $(x, y)$  and  $(\xi, \eta)$  wing coordinates (see Figure 8) can be written as

$$\xi = \frac{2L(x - y \tan \theta)}{C_R L - (C_R - C_t)y} - 1 \quad (30)$$

$$\eta = \frac{2y}{L} - 1 \quad (31)$$

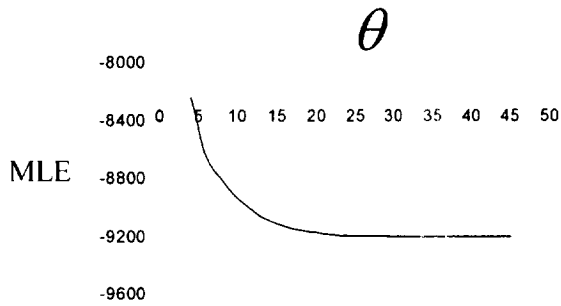
where  $-1 \leq \xi \leq 1$ , and  $-1 \leq \eta \leq 1$



**Figure 9. Finite Element Model for Stress Prediction**

The four corner node thicknesses ( $t_1$  to  $t_4$ ) are the design variables. Each thickness is allowed to vary between  $0.15 \text{ in}$  and  $0.4 \text{ in}$ . The sampling points are generated using the PLS scheme for level-5 and level-7. There are 97 PLS sampling points for level-5 and 881 PLS sampling points for level-7. In order to predict the stress distribution as function of four design variables ( $t_1$  to  $t_4$ ), the plate is divided into 162 quadrilateral finite elements as shown in Figure 9. Finite element analyses with 162 quadrilateral elements are used to obtain the stresses at the centroids of each element. For example, for the 881 sampling points in level-7, 881 finite element analyses are performed. These 881 centroidal stresses for an element are used to construct the responses for that element. Hence, total of 162 response surfaces are constructed, one for each element. The NASTRAN structural analysis code with 8-node quadrilateral elements is used for the finite element analyses.

The 162 response surfaces (one for each finite element) are generated using GLS, kriging, and MLS methods in terms of the design variables  $t_1$  to  $t_4$ . The fitting of the response surface with GLS and MLS is straightforward, but, the kriging method requires some clarification. As previously mentioned, the estimation of  $\theta$  in Equation (4) using one-dimensional optimization is the critical step in kriging. The variation of Maximum Likelihood Estimate (MLE) with  $\theta$  is shown in Figure 10.



**Figure 10. Variation of Maximum Likelihood Estimate (MLE) with Free Parameter  $\theta$**

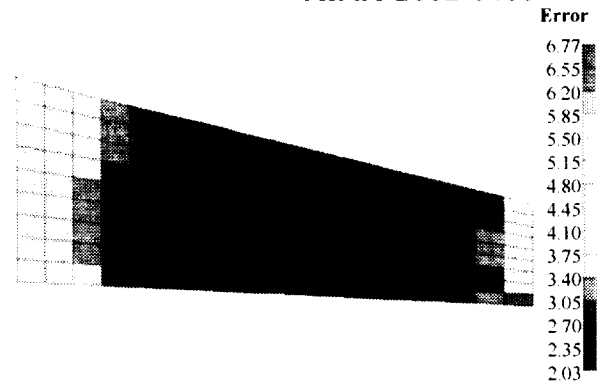
It can be seen from Figure 10 that there is no absolute maximum value of MLE. The MLE reaches the maximum value at the default minimum value of  $\theta = 0$  degrees. The matrix  $R$  in Equation (3) becomes singular for  $\theta = 0$  degrees. For the current problem with 881 sample points, the determinant of the matrix  $R$  becomes numerically nearly zero for  $\theta \leq 3$ . The selection of  $\theta$  here then becomes highly subjective. The value of  $\theta$  is set to  $\theta = 7$  and  $\theta = 5$  for level-5 and level-7 respectively. These  $\theta$  values are selected, since they produced accurate results.

To examine the fitting performance, 2500 randomly selected thickness sets are used in Equation (23) to obtain the average error. The average errors are calculated for all the 162 elements. The stress values obtained from the 2500 NASTRAN analyses are used as 'exact' in Equation (23). The average error for each element is normalized by the average stress for that element using

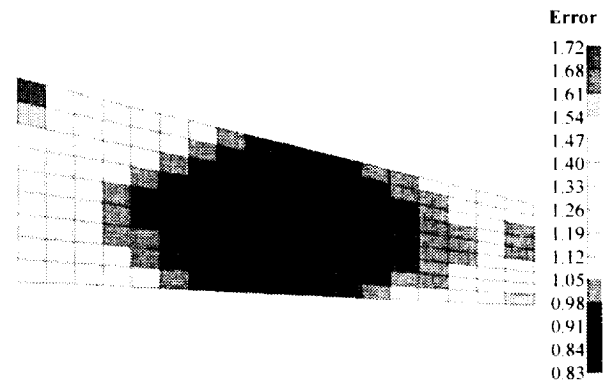
$$\text{Mean Stress for an element} = \frac{\sum_{j=1}^{2500} |\text{exact stress}|}{2500} \quad (32)$$

The percent error is calculated as

$$\% \text{Error for an element} = \frac{\text{Average Error}}{\text{Mean Stress}} \times 100 \quad (33)$$



**Figure 11. Average Error Distribution for GLS Method Over the Wing Span**



**Figure 12. Average Error Distribution for Kriging**



**Method Over the Wing Span**

**Figure 13. Average Error Distribution for MLS Method Over the Wing Span**

The distributions of average errors over the wingspan are shown for GLS, kriging and MLS methods in Figures 11 to 13. The relative differences in average error for the three methods are almost the same in each element. As a representative example, the percent error in Element 1 located near the root of the wing (see Figure 9) is compared for the three interpolation methods in Figures 14 and 15 for levels 5 and 7. For level-5 with 91 points, the GLS produced 8.65 percent error compared to 1.8 percent for kriging and 1.5 percent error for MLS. The corresponding values for level-7 with 881 points are GLS=5.8 percent error, kriging=1.7 percent error and MLS=0.58 percent error. The kriging and MLS methods produce errors that are an order of magnitude less than the GLS method. Numerically the MLS produced the least error at both levels. However, between kriging and MLS, it is difficult to choose one over the other. Further study is needed to draw definite conclusions.

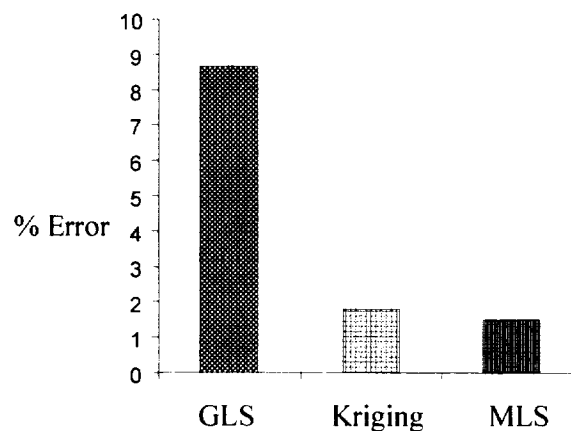
### Discussion

In the above two examples it was shown that the performance of the kriging and MLS methods are nearly the same. However, in kriging there is a need to estimate the free parameter  $\theta$  through optimization. In the MLS method there is no parameter to evaluate, except to define the influence radius  $l_i$ . The free parameter  $l_i$  is easy to select from the requirement of the number of points to make the matrix  $A$  in Equation (15) non-singular. It should be noted that the kriging and the MLS methods are called locally weighted interpolation rather than piecewise interpolation. It was observed in the two variables example that the piecewise polynomials produced small error at all the levels. But these methods are available only for two variables. Hence, piecewise polynomial methods for arbitrary number of dimensions would need to be developed.

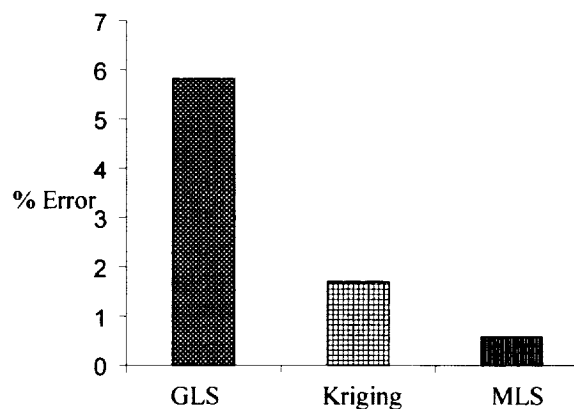
### Concluding Remarks

The usefulness of Piecewise Polynomials with higher order derivative continuity methods are studied and evaluated for response surface construction. The piecewise polynomials with higher order derivative continuity ( $C^1$  and  $C^2$  continuity) methods produce fewer errors for a given set of sample points than the global least squares methods. The kriging and MLS methods performed equally well and are easy to extend to arbitrary numbers of design variables. However, it is

easier to fix the free parameters in the MLS method than in the kriging method. There is a need to develop or examine piecewise polynomial methods that can be easily extended to arbitrary number of design variables.



**Figure 14 . Average Errors: Four-Variables Problem With 97 Sampling Points**



**Figure 15. Average Errors: Four-Variables Problem With 881 Sampling Points**

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### References

1. Stroud, W. Jefferson; Krishnamurthy, T.; and Smith, Steven A., "Probabilistic and Possibilistic Analyses of the Strength of a Bonded", Joint. Presented at 42<sup>nd</sup> AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, AIAA Paper No. 2001-1238, April 16-19, 2001, Seattle, WA. Available on CDROM Vol 6, No. 2, AIAA, Reston, VA.
2. Sundararajan, C., "Probabilistic Structural Mechanics Handbook". Chapman & Hall, 1995.
3. Myers, H., R., and Montgomery, C., D., "Response Surface Methodology", a Wiley-Interscience Publication. John Wiley & Sons, Inc., 1995.
4. Romero, V. J., Swiler, L.P., and Giunta, A.A., "Application of Finite Element, Global Polynomial, and Kriging Response Surfaces in Progressive Lattice Sampling Designs", 8<sup>th</sup> ASCE Specialty Conference on Probabilistic Mechanics and Structural Reliability, PMC 2000-175, 2000.
5. Romero, V.J., and Bankston, S.D., "Finite-Element/Progressive-Lattice-Sampling Response Surface Methodology and Application to Benchmark Probability Quantification Problems", Sandia National Laboratories report SAND98-0567, 1998.
6. Nielson, G., "Minimum Norm Interpolation in Triangle", *SIAM J. Numer. Anal.* Vol. 17, pp. 44-61, 1980.
7. Nielson, G., "The Side-Vertex Method for Interpolation in Triangle", *Jl. of Approximation Theory*, Vol. 25, pp. 318-336, 1979.
8. Pottmann, H., "Scattered Data Interpolation Based upon Generalized Minimum Norm Networks", *Constructive Approximation*, Vol. 7, pp. 247-256, 1991.
9. Alfeld, P., and Barnhill, R. E., "A transfinite  $C^2$  Interpolant over Triangles", *Rocky Mountain Journal of Mathematics*, Vol. 14, pp. 17-39, 1984.
10. Aminpour, M.A., Krishnamurthy, T. and Fadale, T.D., "Coupling Independently Modeled Three-Dimensional Finite Element Meshes with Arbitrary Shape Interface Boundaries," *Proceedings of 39th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, Long Beach, CA, Part 4, pp. 3014-3024, 1998.
11. Stroud, W. Jefferson, Krishnamurthy, T., Mason, B. H., Smith, S. A., and Naser, S. Ahmad., "Probabilistic Design of a Plate-Like Wing to Meet Flutter and Strength Requirements", To be Presented at 43<sup>rd</sup> AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, AIAA Paper No. 2002-1464, April 22-25, Denver, CO, 2002.
12. Giunta, Anthony A., and Watson, Layne T., "A Comparison of Approximation Modeling Techniques: Polynomial versus Interpolating Models", paper AIAA-98-4758 presented at the 7<sup>th</sup> AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, St. Louis, MO, Sept. 2-4, 1998.
13. Atluri, S N., and Zhu, T., "A new meshless Local Petrov-Galerkin (MLPG) approach in Computational Mechanics", *Computational Mechanics*, Vol. 22, pp. 117-127, 1998.
14. Raju, I. S., and Chen, T., "Meshless Petrov-Galerkin Method Applied To Axisymmetric Problems", 42<sup>nd</sup> AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, AIAA Paper No. 2001-1253, April 16-19, Seattle, WA, 2001.
15. Choi, K.K. Youn, B., and Yang, R-J. "Moving Least Squares Method for Reliability-Based Design Optimization." *Fourth World Congress of Structural and Multidisciplinary Optimization*, Dalian, China, June 4-8, 2001.

